Error Analysis of a Simple Quaternion Estimator: the Gaussian Case

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ABSTRACT

Gaussian noise is introduced when analyzing the error characteristics of a novel single-frame quaternion estimator, establishing the fourth-order approximate covariance matrix expression to address normalized additive errors. In contrast to the second-order approximations, these expressions not only provide increased precision but also alleviate issues related to singularity. This paper presents a comprehensive derivation of the individual components within the analytical expressions, all conducted within the framework of four-dimensional algebra.

Keywords: Attitude quaternion, fourth-order covariance analysis, Gaussian noise

1 Introduction

The performance of attitude estimators for spacecraft is of paramount importance, as it directly impacts the success or failure of missions. This task faces the continuous challenge of increasing noise stemming from various sources, including sensors, environmental factors, and mechanical vibrations. These noises are inevitable and introduce non-negligible errors in the estimation process.

Gaussian noise, commonly referred to as white noise, is a fundamental concept in the fields of statistics and signal processing, permeating various scientific and engineering applications. It represents a form of random variation with a distribution following the classic curve of the Gaussian (normal) distribution. The hallmark of this noise is its zero mean, with the spread of its values determined by the standard deviation. This fundamental concept plays a pivotal role in modern science because it simulates the randomness and uncertainty inherent in many complex processes in both the natural world and technological systems.

Based on the simple pose estimator proposed in [1] and its error analysis, our focus is on the covariance matrix of the estimated error. This is a crucial task because this matrix measures our confidence in the estimate. To address the interference caused by noise, we need to develop more accurate and reliable numerical expressions. When constructing the normalized additive fourth-order approximation covariance matrix, it is assumed that both the measurement and observation vectors are affected by Gaussian noise. This numerical expression is more accurate than the traditional second-order approximation result, while avoiding the trap of mathematical singularities.

2 Preliminaries

This section follows [1] , the additive quaternion error and bias expression are as follows:

$$\Delta\widehat{\mathbf{q}} = N_1\Delta\check{\mathbf{q}} + N_2\mathbf{q} \tag{1}$$

$$E\{\Delta\widehat{\mathbf{q}}\} = \bar{N}_2\mathbf{q} \tag{2}$$

$$\Delta\widehat{\mathbf{q}} - E\{\Delta\widehat{\mathbf{q}}\} = N_1\Delta\check{\mathbf{q}} + (N_2 - \bar{N}_2)\mathbf{q} \tag{3}$$

where

$$N_1 = I_4 - \mathbf{q}\mathbf{q}^T \tag{4}$$

$$N_2 = \Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T + \frac{1}{2}\operatorname{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\, I_4 \tag{5}$$

$$\bar{N}_2 = P_{\Delta\check{\mathbf{q}}} + \frac{1}{2}\operatorname{tr}(MP_{\Delta\check{\mathbf{q}}})\, I_4 \tag{6}$$

$$M = I_4 - 3\mathbf{q}\mathbf{q}^T \tag{7}$$

The covariance of the normalized quaternion additive error is as follows and the derivation of Eq.(8) is provided in Appendix A :

$$\begin{aligned}
P_{\Delta\widehat{\mathbf{q}}} =& N_1 P_{\Delta\check{\mathbf{q}}} N_1^T - E\{P_{\Delta\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^T\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} - \frac{1}{2}E\{P_{\Delta\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^T[\Delta\check{\mathbf{q}}^T M\Delta\check{\mathbf{q}} I_4]\} \\
& - \frac{1}{2}E\{\operatorname{tr}(MP_{\Delta\check{\mathbf{q}}})\, I_4\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\Delta\check{\mathbf{q}}\} - \frac{1}{4}E\{\operatorname{tr}(MP_{\Delta\check{\mathbf{q}}})\, I_4\Delta\check{\mathbf{q}}^T\mathbf{q}\operatorname{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\, I_4\} \\
& + E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2}E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\operatorname{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\, I_4\} \\
& + \frac{1}{2}E\{\operatorname{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\, I_4\mathbf{q}\mathbf{q}^T\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{4}E\{\operatorname{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\, I_4\mathbf{q}\mathbf{q}^T\operatorname{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\, I_4\}
\end{aligned} \tag{8}$$

The fourth-order terms in $P_{\Delta\widehat{\mathbf{q}}}$ can be written as follows and the derivation of Eq.(9) is also provided in Appendix A:

$$\begin{aligned}
P_{\Delta\widehat{\mathbf{q}}4th} =& + E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} - 3E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^3\Delta\check{\mathbf{q}}\mathbf{q}^T\} + E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\} \\
& + \frac{1}{4}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})^2\mathbf{q}\mathbf{q}^T\} - \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\mathbf{q}\mathbf{q}^T\} + \frac{9}{4}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^4\mathbf{q}\mathbf{q}^T\}
\end{aligned} \tag{9}$$

3 Gaussian noise case

Let $\mathbf{q} = [q_1, q_2, q_3, q_4]^T$, and $\Delta\check{\mathbf{q}} = [x_1, x_2, x_3, x_4]^T$, $\{x_1, x_2, x_3, x_4\}$ are zero-mean jointly Gaussian random variables and the standard deviation is σ_{ij}:

$$x_i \sim N(0, \sigma_{ij})$$

The joint moment [2] is:

$$E\{x_i x_j x_k x_m\} = E\{x_i x_j\}E\{x_k x_m\} + E\{x_i x_k\}E_{\{x_j x_m\}} + E\{x_i x_m\}E\{x_j x_k\} \tag{10}$$

$$E\{x_i x_j x_k x_m\} = \sigma_{ij}\sigma_{km} + \sigma_{ik}\sigma_{jm} + \sigma_{im}\sigma_{jk} \tag{11}$$

$$E\{x_i^2 x_j x_k\} = \sigma_i^2\sigma_{jk} + 2\sigma_{ij}\sigma_{ik} \tag{12}$$

$$E\{x_i^2 x_j^2\} = \sigma_i^2\sigma_j^2 + 2\sigma_{ij}^2 \tag{13}$$

$$E\{x_i^3 x_j\} = 3\sigma_i^2\sigma_{ij} \tag{14}$$

$$E\{x_i^4\} = 3\sigma_i^4 \tag{15}$$

where $P_{\Delta\check{\mathbf{q}}} = \frac{1}{|\bar{\mathbf{q}}^t|^2} P_{\Delta\bar{\mathbf{q}}}$ and $i, j, k, m = 1, 2, 3, 4$.

Then derive a numerical expression for each term in Eq.(9), the proofs of Eqs. (16)-(21) are provided in Appendix B.

There is the expectation matrix for term 1:

$$E\{(\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q})^2\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\} \tag{16}$$

$$= \begin{pmatrix}
\begin{array}{l} 3q_1^2\sigma_1^4 + q_2^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\ + 6q_1q_2\sigma_1^2\sigma_{12} + q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\ + 6q_1q_3\sigma_1^2\sigma_{13} + 6q_1q_4\sigma_1^2\sigma_{14} \\ + q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\ + 2q_2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\ + 2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \end{array} &
\begin{array}{l} 3q_1^2\sigma_1^2\sigma_{12} + 3q_2^2\sigma_2^2\sigma_{12} \\ + 2q_1q_2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\ + q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\ + 2q_1q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\ + 2q_2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\ + q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ + 2q_1q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 2q_2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} &
\begin{array}{l} 3q_1^2\sigma_1^2\sigma_{13} + 3q_3^2\sigma_3^2\sigma_{13} \\ + q_2^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\ + 2q_1q_2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\ + 2q_1q_3(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\ + 2q_2q_3(\sigma_{12}\sigma_3^2 + 2\sigma_{13}\sigma_{23}) \\ + q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\ + 2q_1q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ + 2q_3q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{14}) \\ + 2q_2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} &
\begin{array}{l} 3q_1^2\sigma_1^2\sigma_{14} + 3q_4^2\sigma_4^2\sigma_{14} \\ + q_2^2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_1q_2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 2q_1q_4(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\ + 2q_2q_4(\sigma_{12}\sigma_4^2 + 2\sigma_{14}\sigma_{24}) \\ + q_3^2(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ + 2q_1q_3(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ + 2q_3q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_2q_3(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} \\
\begin{array}{l} 3q_1^2\sigma_1^2\sigma_{12} + 3q_2^2\sigma_2^2\sigma_{12} \\ + 2q_1q_2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\ + q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\ + 2q_1q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\ + 2q_2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\ + q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ + 2q_1q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 2q_2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} &
\begin{array}{l} 3q_2^2\sigma_2^4 + q_1^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\ + 6q_1q_2\sigma_2^2\sigma_{12} + 6q_2q_4\sigma_2^2\sigma_{24} \\ + q_3^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\ + 6q_2q_3\sigma_2^2\sigma_{23} \\ + 2q_1q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\ + q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\ + 2q_1q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_3q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \end{array} &
\begin{array}{l} q_1^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\ + 3q_2^2\sigma_2^2\sigma_{23} + 3q_3^2\sigma_3^2\sigma_{32} \\ + 2q_1q_2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\ + 2q_1q_3(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\ + 2q_2q_3(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\ + q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_2q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_3q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\ + 2q_1q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} &
\begin{array}{l} q_1^2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 3q_2^2\sigma_2^2\sigma_{23} + 3q_4^2\sigma_4^2\sigma_{24} \\ + 2q_1q_2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_1q_4(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ + 2q_2q_4(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\ + q_3^2(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\ + 2q_2q_3(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_3q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_1q_3(\sigma_{12}\sigma_{34} + \sigma_{14}\sigma_{23} + \sigma_{13}\sigma_{24}) \end{array} \\
\begin{array}{l} 3q_1^2\sigma_1^2\sigma_{13} + 3q_3^2\sigma_3^2\sigma_{13} \\ + q_2^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\ + 2q_1q_2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\ + 2q_1q_3(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\ + 2q_2q_3(\sigma_{12}\sigma_3^2 + 2\sigma_{13}\sigma_{23}) \\ + q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\ + 2q_1q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ + 2q_3q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{14}) \\ + 2q_2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} &
\begin{array}{l} q_1^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\ + 3q_2^2\sigma_2^2\sigma_{23} + 3q_3^2\sigma_3^2\sigma_{32} \\ + 2q_1q_2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\ + 2q_1q_3(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\ + 2q_2q_3(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\ + q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_2q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_3q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\ + 2q_1q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} &
\begin{array}{l} 3q_3^2\sigma_3^4 + 6q_2q_3\sigma_3^2\sigma_{23} \\ + q_2^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\ + q_1^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\ + 6q_1q_3\sigma_3^2\sigma_{13} + 6q_3q_4\sigma_3^2\sigma_{34} \\ + 2q_1q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ + q_4^2(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\ + 2q_2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\ + 2q_1q_2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \end{array} &
\begin{array}{l} q_1^2(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ + 3q_3^2\sigma_3^2\sigma_{34} + 3q_4^2\sigma_4^2\sigma_{43} \\ + 2q_1q_3(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ + 2q_2q_3(\sigma_3^2\sigma_{42} + 2\sigma_{23}\sigma_{34}) \\ + 2q_3q_4(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\ + q_2^2(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_2q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_1q_4(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\ + 2q_1q_2(\sigma_{12}\sigma_{34} + \sigma_{14}\sigma_{23} + \sigma_{13}\sigma_{24}) \end{array} \\
\begin{array}{l} 3q_1^2\sigma_1^2\sigma_{14} + 3q_4^2\sigma_4^2\sigma_{14} \\ + q_2^2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_1q_2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 2q_1q_4(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\ + 2q_2q_4(\sigma_{12}\sigma_4^2 + 2\sigma_{14}\sigma_{24}) \\ + q_3^2(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ + 2q_1q_3(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ + 2q_3q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_2q_3(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}) \end{array} &
\begin{array}{l} q_1^2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\ + 3q_2^2\sigma_2^2\sigma_{23} + 3q_4^2\sigma_4^2\sigma_{24} \\ + 2q_1q_2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\ + 2q_1q_4(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ + 2q_2q_4(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\ + q_3^2(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\ + 2q_2q_3(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_3q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_1q_3(\sigma_{12}\sigma_{34} + \sigma_{14}\sigma_{23} + \sigma_{13}\sigma_{24}) \end{array} &
\begin{array}{l} q_1^2(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\ + 3q_3^2\sigma_3^2\sigma_{34} + 3q_4^2\sigma_4^2\sigma_{43} \\ + 2q_1q_3(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\ + 2q_2q_3(\sigma_3^2\sigma_{42} + 2\sigma_{23}\sigma_{34}) \\ + 2q_3q_4(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\ + q_2^2(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\ + 2q_2q_4(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_1q_4(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\ + 2q_1q_2(\sigma_{12}\sigma_{34} + \sigma_{14}\sigma_{23} + \sigma_{13}\sigma_{24}) \end{array} &
\begin{array}{l} 3q_4^2\sigma_4^4 + 6q_2q_4\sigma_4^2\sigma_{24} \\ + q_2^2(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\ + q_1^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\ + 6q_1q_4\sigma_4^2\sigma_{14} + 6q_3q_4\sigma_4^2\sigma_{34} \\ + 2q_1q_2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\ + q_3^2(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\ + 2q_2q_3(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\ + 2q_1q_3(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \end{array}
\end{pmatrix}$$

There is the expectation matrix for term 2:

$$-3E\{(\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q})^3\boldsymbol{\Delta}\check{\mathbf{q}}\mathbf{q}^T\} \tag{17}$$

$$= -3\begin{pmatrix}
\begin{array}{l}
3q_1^4\sigma_1^4 + 3q_1q_2^3\sigma_2^2\sigma_{12} + 9q_1^3q_4\sigma_1^2\sigma_{14} \\
+ 3q_1^2q_2^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 9q_1^3q_2\sigma_1^2\sigma_{12} + 3q_1q_3^3\sigma_3^2\sigma_{13} \\
+ 3q_1^2q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\
+ 3q_1q_2q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 9q_1^3q_3\sigma_1^2\sigma_{13} + 3q_1q_4^3\sigma_4^2\sigma_{14} \\
+ 3q_1q_2^2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 6q_1^2q_2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 3q_1^2q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 3q_1q_2q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_1q_3q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 3q_1q_2^2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 6q_1^2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_1q_3^2q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 6q_1^2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 6q_1q_2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array} &
\begin{array}{l}
3q_2^4\sigma_2^2\sigma_{12} + 3q_1q_2^3(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 9q_1^2q_2^2\sigma_1^2\sigma_{12} + 9q_1^2q_2q_4\sigma_1^2\sigma_{14} \\
+ 3q_1^3q_2\sigma_1^4 + 3q_2q_3^3\sigma_3^2\sigma_{13} \\
+ 3q_2^2q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 3q_1q_2q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\
+ 3q_2^3q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 6q_1q_2^2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 9q_1^2q_2q_3\sigma_1^2\sigma_{13} + 3q_2q_4^3\sigma_4^2\sigma_{14} \\
+ 3q_2^2q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_1q_2q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 3q_2q_3q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 3q_2^3q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 6q_1q_2^2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_2q_3^2q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 6q_2^2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 6q_1q_2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14})
\end{array} &
\begin{array}{l}
3q_3^4\sigma_3^2\sigma_{13} + 9q_1^2q_3q_4\sigma_1^2\sigma_{14} \\
+ 3q_1q_3^3(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) + 3q_1^3q_3\sigma_1^4 \\
+ 9q_1^2q_3^2\sigma_1^2\sigma_{13} + 3q_2^3q_3\sigma_2^2\sigma_{12} \\
+ 3q_2^2q_3^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 3q_1q_2^2q_3(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3^3q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 6q_1q_2q_3^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 9q_1^2q_2q_3\sigma_1^2\sigma_{12} + 3q_3q_4^3\sigma_4^2\sigma_{14} \\
+ 3q_3^2q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 3q_1q_3q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 3q_2q_3q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_2^2q_3q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 6q_1q_2q_3q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 6q_1q_3^2q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 6q_2q_3^2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 3q_2q_3^3(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23})
\end{array} &
\begin{array}{l}
3q_4^4\sigma_4^2\sigma_{14} + 3q_1q_4^3(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 9q_1^2q_4^2\sigma_1^2\sigma_{14} + 9q_1^2q_3q_4\sigma_1^2\sigma_{13} \\
+ 3q_1^3q_4\sigma_1^4 + 3q_2^3q_4\sigma_2^2\sigma_{12} \\
+ 3q_2^2q_4^2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 3q_1q_2^2q_4(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3q_4^3(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 6q_1q_2q_4^2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 9q_1^2q_2q_4\sigma_1^2\sigma_{12} + 3q_3^3q_4\sigma_3^2\sigma_{13} \\
+ 3q_3^2q_4^2(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 3q_1q_3^2q_4(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\
+ 3q_2q_3^2q_4(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 3q_2^2q_3q_4(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 6q_1q_2q_3q_4(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 6q_1q_3q_4^2(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 6q_2q_3q_4^2(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 3q_2q_4^3(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24})
\end{array} \\
\begin{array}{l}
3q_2^4\sigma_2^2\sigma_{12} + 3q_1q_2^3(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 9q_1^2q_2^2\sigma_1^2\sigma_{12} + 9q_1^2q_2q_4\sigma_1^2\sigma_{14} \\
+ 3q_1^3q_2\sigma_1^4 + 3q_2q_3^3\sigma_3^2\sigma_{13} \\
+ 3q_2^2q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 3q_1q_2q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\
+ 3q_2^3q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 6q_1q_2^2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 9q_1^2q_2q_3\sigma_1^2\sigma_{13} + 3q_2q_4^3\sigma_4^2\sigma_{14} \\
+ 3q_2^2q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_1q_2q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 3q_2q_3q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 3q_2^3q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 6q_1q_2^2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_2q_3^2q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 6q_2^2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 6q_1q_2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14})
\end{array} &
\begin{array}{l}
+ 3q_2q_3^3\sigma_3^2\sigma_{23} + 9q_2^3q_4\sigma_2^2\sigma_{24} \\
3q_2^4\sigma_2^4 + 3q_1^2q_2^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 9q_1q_2^3\sigma_2^2\sigma_{12} + 3q_1^3q_2\sigma_1^2\sigma_{12} \\
+ 3q_2^2q_3^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 3q_1q_2q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 9q_2^3q_3\sigma_2^2\sigma_{23} + 3q_2q_4^3\sigma_4^2\sigma_{24} \\
+ 6q_1q_2^2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 3q_1^2q_2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 3q_2^2q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\
+ 3q_1q_2q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_2q_3q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\
+ 6q_1q_2^2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 3q_1^2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_2q_3^2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\
+ 9q_2^2q_3q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 6q_1q_2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array} &
\begin{array}{l}
3q_3^4\sigma_3^2\sigma_{23} + 9q_2^2q_3^2\sigma_2^2\sigma_{23} \\
+ 3q_1q_3^3(\sigma_{13}\sigma_3^2 + 2\sigma_{13}\sigma_{23}) \\
+ 9q_2^2q_3q_4\sigma_2^2\sigma_{24} + 3q_1^3q_3\sigma_1^2\sigma_{12} \\
+ 3q_3^2q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\
+ 3q_1^2q_2q_3(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3^3q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\
+ 6q_1q_2q_3^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 9q_1q_2^2q_3\sigma_2^2\sigma_{12} + 3q_3q_4^3\sigma_4^2\sigma_{24} \\
+ 3q_1^2q_3^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_1q_3q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_2q_3q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{12}^2) \\
+ 6q_1q_2q_3q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 3q_2^3q_3\sigma_2^4 + 3q_2q_3^3(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 6q_2q_3^2q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 6q_1q_3^2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array} &
\begin{array}{l}
3q_4^4\sigma_4^2\sigma_{24} + 9q_2^2q_3q_4\sigma_2^2\sigma_{23} \\
+ 3q_1q_4^3(\sigma_{12}\sigma_4^2 + 2\sigma_{14}\sigma_{24}) + 3q_2^3q_4\sigma_2^4 \\
+ 9q_2^2q_4^2\sigma_2^2\sigma_{24} + 3q_1^3q_4\sigma_1^2\sigma_{12} \\
+ 3q_3^2q_4^2(\sigma_{24}\sigma_3^2 + 2\sigma_{23}\sigma_{34}) \\
+ 3q_1^2q_2q_4(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3q_4^3(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\
+ 6q_1q_2q_4^2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 9q_1q_2^2q_4\sigma_2^2\sigma_{12} + 3q_3^3q_4\sigma_3^2\sigma_{23} \\
+ 3q_1^2q_4^2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 6q_2q_3q_4^2(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 3q_2q_3^2q_4(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 6q_1q_2q_3q_4(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 3q_2q_4^3(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\
+ 3q_1q_3^2q_4(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 6q_1q_3q_4^2(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array} \\
\begin{array}{l}
3q_3^4\sigma_3^2\sigma_{13} + 9q_1^2q_3q_4\sigma_1^2\sigma_{14} \\
+ 3q_1q_3^3(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) + 3q_1^3q_3\sigma_1^4 \\
+ 9q_1^2q_3^2\sigma_1^2\sigma_{13} + 3q_2^3q_3\sigma_2^2\sigma_{12} \\
+ 3q_2^2q_3^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 3q_1q_2^2q_3(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3^3q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 6q_1q_2q_3^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 9q_1^2q_2q_3\sigma_1^2\sigma_{12} + 3q_3q_4^3\sigma_4^2\sigma_{14} \\
+ 3q_3^2q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 3q_1q_3q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 3q_2q_3q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_2^2q_3q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 6q_1q_2q_3q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 6q_1q_3^2q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 6q_2q_3^2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 3q_2q_3^3(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23})
\end{array} &
\begin{array}{l}
3q_3^4\sigma_3^2\sigma_{23} + 9q_2^2q_3^2\sigma_2^2\sigma_{23} \\
+ 3q_1q_3^3(\sigma_{13}\sigma_3^2 + 2\sigma_{13}\sigma_{23}) \\
+ 9q_2^2q_3q_4\sigma_2^2\sigma_{24} + 3q_1^3q_3\sigma_1^2\sigma_{12} \\
+ 3q_3^2q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\
+ 3q_1^2q_2q_3(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3^3q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\
+ 6q_1q_2q_3^2(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 9q_1q_2^2q_3\sigma_2^2\sigma_{12} + 3q_3q_4^3\sigma_4^2\sigma_{24} \\
+ 3q_1^2q_3^2(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_1q_3q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_2q_3q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{12}^2) \\
+ 6q_1q_2q_3q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 3q_2^3q_3\sigma_2^4 + 3q_2q_3^3(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 6q_2q_3^2q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 6q_1q_3^2q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array} &
\begin{array}{l}
3q_3^4\sigma_3^4 + 2\sigma_{13}^2) + 9q_3^3q_4\sigma_3^2\sigma_{34} \\
+ 3q_1^2q_3^2(\sigma_1^2\sigma_3^2 + 9q_2q_3^3\sigma_3^2\sigma_{23} \\
+ 9q_1q_3^3\sigma_3^2\sigma_{13} + 3q_1^3q_3\sigma_1^2\sigma_{13} \\
+ 3q_2^2q_3^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 6q_1q_2q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 3q_2^3q_3\sigma_2^2\sigma_{23} + 3q_3q_4^3\sigma_4^2\sigma_{34} \\
+ 3q_1q_2^2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 3q_1^2q_2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 3q_3^2q_4^2(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\
+ 3q_1q_3q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 3q_2q_3q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\
+ 6q_1q_3^2q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 3q_2^2q_3q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 6q_2q_3^2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\
+ 6q_1q_2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array} &
\begin{array}{l}
3q_4^4\sigma_4^2\sigma_{34} + 3q_2^3q_4\sigma_2^2\sigma_{23} + 3q_3^3q_4\sigma_3^4 \\
+ 3q_1q_4^3(\sigma_{13}\sigma_4^2 + 2\sigma_{14}\sigma_{34}) \\
+ 9q_3^2q_4^2\sigma_3^2\sigma_{34} + 3q_1^3q_4\sigma_1^2\sigma_{13} \\
+ 3q_1^2q_4^2(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\
+ 3q_3q_4^3(\sigma_4^2\sigma_3^2 + 2\sigma_{34}^2) \\
+ 3q_2^2q_4^2(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 9q_1q_3^2q_4\sigma_3^2\sigma_{13} + 9q_2q_3^2q_4\sigma_3^2\sigma_{23} \\
+ 3q_1^2q_2q_4(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 6q_1q_3q_4^2(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 6q_2^2q_3q_4(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 3q_2q_3q_4^2(\sigma_{24}\sigma_3^2 + 2\sigma_{23}\sigma_{34}) \\
+ 6q_1q_2q_3q_4(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 3q_1q_2^2q_4(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 6q_1q_2q_4^2(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 3q_2q_4^3(\sigma_{23}\sigma_4^2 + 2\sigma_{24}\sigma_{34})
\end{array} \\
\begin{array}{l}
3q_4^4\sigma_4^2\sigma_{14} + 3q_1q_4^3(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 9q_1^2q_4^2\sigma_1^2\sigma_{14} + 9q_1^2q_3q_4\sigma_1^2\sigma_{13} \\
+ 3q_1^3q_4\sigma_1^4 + 3q_2^3q_4\sigma_2^2\sigma_{12} \\
+ 3q_2^2q_4^2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 3q_1q_2^2q_4(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3q_4^3(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 6q_1q_2q_4^2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 9q_1^2q_2q_4\sigma_1^2\sigma_{12} + 3q_3^3q_4\sigma_3^2\sigma_{13} \\
+ 3q_3^2q_4^2(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 3q_1q_3^2q_4(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\
+ 3q_2q_3^2q_4(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 3q_2^2q_3q_4(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 6q_1q_2q_3q_4(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 6q_1q_3q_4^2(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 6q_2q_3q_4^2(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 3q_2q_4^3(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24})
\end{array} &
\begin{array}{l}
3q_4^4\sigma_4^2\sigma_{24} + 9q_2^2q_3q_4\sigma_2^2\sigma_{23} \\
+ 3q_1q_4^3(\sigma_{12}\sigma_4^2 + 2\sigma_{14}\sigma_{24}) + 3q_2^3q_4\sigma_2^4 \\
+ 9q_2^2q_4^2\sigma_2^2\sigma_{24} + 3q_1^3q_4\sigma_1^2\sigma_{12} \\
+ 3q_3^2q_4^2(\sigma_{24}\sigma_3^2 + 2\sigma_{23}\sigma_{34}) \\
+ 3q_1^2q_2q_4(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) \\
+ 3q_3q_4^3(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\
+ 6q_1q_2q_4^2(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 9q_1q_2^2q_4\sigma_2^2\sigma_{12} + 3q_3^3q_4\sigma_3^2\sigma_{23} \\
+ 3q_1^2q_4^2(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 6q_2q_3q_4^2(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 3q_2q_3^2q_4(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 6q_1q_2q_3q_4(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 3q_2q_4^3(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\
+ 3q_1q_3^2q_4(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 6q_1q_3q_4^2(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array} &
\begin{array}{l}
3q_4^4\sigma_4^2\sigma_{34} + 3q_2^3q_4\sigma_2^2\sigma_{23} + 3q_3^3q_4\sigma_3^4 \\
+ 3q_1q_4^3(\sigma_{13}\sigma_4^2 + 2\sigma_{14}\sigma_{34}) \\
+ 9q_3^2q_4^2\sigma_3^2\sigma_{34} + 3q_1^3q_4\sigma_1^2\sigma_{13} \\
+ 3q_1^2q_4^2(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) \\
+ 3q_3q_4^3(\sigma_4^2\sigma_3^2 + 2\sigma_{34}^2) \\
+ 3q_2^2q_4^2(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 9q_1q_3^2q_4\sigma_3^2\sigma_{13} + 9q_2q_3^2q_4\sigma_3^2\sigma_{23} \\
+ 3q_1^2q_2q_4(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) \\
+ 6q_1q_3q_4^2(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 6q_2^2q_3q_4(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) \\
+ 3q_2q_3q_4^2(\sigma_{24}\sigma_3^2 + 2\sigma_{23}\sigma_{34}) \\
+ 6q_1q_2q_3q_4(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) \\
+ 3q_1q_2^2q_4(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) \\
+ 6q_1q_2q_4^2(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34}) \\
+ 3q_2q_4^3(\sigma_{23}\sigma_4^2 + 2\sigma_{24}\sigma_{34})
\end{array} &
\begin{array}{l}
3q_4^4\sigma_4^4 + 9q_1q_4^3\sigma_4^2\sigma_{14} + 3q_3^3q_4\sigma_3^2\sigma_{34} \\
+ 9q_2q_4^3\sigma_4^2\sigma_{24} + 9q_3q_4^3\sigma_4^2\sigma_{34} \\
+ 3q_1^2q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) \\
+ 3q_2^2q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) \\
+ 6q_1q_2q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
+ 3q_3^2q_4^2(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) \\
+ 6q_1q_3q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) \\
+ 6q_2q_3q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) \\
+ 3q_1^3q_4\sigma_1^2\sigma_{14} + 3q_2^3q_4\sigma_2^2\sigma_{24} \\
+ 3q_1q_2^2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) \\
+ 3q_1^2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) \\
+ 3q_1q_3^2q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
+ 3q_2q_3^2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) \\
+ 3q_1^2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) \\
+ 3q_2^2q_3q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) \\
+ 6q_1q_2q_3q_4(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{12}\sigma_{34})
\end{array}
\end{pmatrix}$$

There is the expectation matrix for term 3:

$$E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\} \tag{18}$$

$$= \begin{pmatrix} 3\sigma_1^4+\sigma_1^2\sigma_2^2+2\sigma_{12}^2 & 3\sigma_1^2\sigma_{12}+3\sigma_2^2\sigma_{12} & 3\sigma_1^2\sigma_{13}+3\sigma_3^2\sigma_{13} & 3\sigma_1^2\sigma_{14}+3\sigma_4^2\sigma_{14} \\ +\sigma_1^2\sigma_3^2+2\sigma_{13}^2 & +\sigma_3^2\sigma_{12}+2\sigma_{13}\sigma_{23} & +\sigma_2^2\sigma_{13}+2\sigma_{12}\sigma_{23} & +\sigma_2^2\sigma_{14}+2\sigma_{12}\sigma_{24} \\ +\sigma_1^2\sigma_4^2+2\sigma_{14}^2 & +\sigma_4^2\sigma_{12}+2\sigma_{14}\sigma_{24} & +\sigma_4^2\sigma_{13}+2\sigma_{14}\sigma_{34} & +\sigma_4^2\sigma_{13}+2\sigma_{14}\sigma_{34} \\ 3\sigma_1^2\sigma_{12}+3\sigma_2^2\sigma_{12} & \sigma_1^2\sigma_2^2+2\sigma_{12}^2+3\sigma_2^4 & \sigma_1^2\sigma_{23}+2\sigma_{12}\sigma_{13} & \sigma_1^2\sigma_{24}+2\sigma_{12}\sigma_{14} \\ +\sigma_3^2\sigma_{12}+2\sigma_{13}\sigma_{23} & +\sigma_2^2\sigma_3^2+2\sigma_{23}^2 & +3\sigma_2^2\sigma_{23}+3\sigma_3^2\sigma_{23} & +3\sigma_2^2\sigma_{24}+3\sigma_4^2\sigma_{24} \\ +\sigma_4^2\sigma_{12}+2\sigma_{14}\sigma_{24} & +\sigma_2^2\sigma_4^2+2\sigma_{24}^2 & +\sigma_4^2\sigma_{23}+2\sigma_{24}\sigma_{34} & +\sigma_3^2\sigma_{24}+2\sigma_{23}\sigma_{34} \\ 3\sigma_1^2\sigma_{13}+3\sigma_3^2\sigma_{13} & \sigma_1^2\sigma_{23}+2\sigma_{12}\sigma_{13} & \sigma_1^2\sigma_3^2+2\sigma_{13}^2+3\sigma_3^4 & \sigma_1^2\sigma_{34}+2\sigma_{13}\sigma_{14} \\ +\sigma_2^2\sigma_{13}+2\sigma_{12}\sigma_{23} & +3\sigma_2^2\sigma_{23}+3\sigma_3^2\sigma_{23} & +\sigma_2^2\sigma_3^2+2\sigma_{23}^2 & +\sigma_2^2\sigma_{34}+2\sigma_{23}\sigma_{24} \\ +\sigma_4^2\sigma_{13}+2\sigma_{14}\sigma_{34} & +\sigma_4^2\sigma_{23}+2\sigma_{24}\sigma_{34} & +\sigma_3^2\sigma_4^2+2\sigma_{34}^2 & +3\sigma_3^2\sigma_{34}+3\sigma_4^2\sigma_{34} \\ 3\sigma_1^2\sigma_{14}+3\sigma_4^2\sigma_{14} & \sigma_1^2\sigma_{24}+2\sigma_{12}\sigma_{14} & \sigma_1^2\sigma_{34}+2\sigma_{13}\sigma_{14} & \sigma_1^2\sigma_4^2+2\sigma_{14}^2+3\sigma_4^4 \\ +\sigma_2^2\sigma_{14}+2\sigma_{12}\sigma_{24} & +3\sigma_2^2\sigma_{24}+3\sigma_4^2\sigma_{24} & +\sigma_2^2\sigma_{34}+2\sigma_{23}\sigma_{24} & +\sigma_2^2\sigma_4^2+2\sigma_{24}^2 \\ +\sigma_4^2\sigma_{13}+2\sigma_{14}\sigma_{34} & +\sigma_3^2\sigma_{24}+2\sigma_{23}\sigma_{34} & +3\sigma_3^2\sigma_{34}+3\sigma_4^2\sigma_{34} & +\sigma_3^2\sigma_4^2+2\sigma_{34}^2 \end{pmatrix} \mathbf{q}\mathbf{q}^T$$

There is the expectation matrix for term 4:

$$\frac{1}{4}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})^2\mathbf{q}\mathbf{q}^T\} \tag{19}$$

$$\begin{aligned} &= \frac{1}{4}[3(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2) + 2(\sigma_1^2\sigma_2^2+2\sigma_{12}^2+\sigma_1^2\sigma_3^2+2\sigma_{13}^2 \\ &+ \sigma_1^2\sigma_4^2+2\sigma_{14}^2+\sigma_3^2\sigma_2^2+2\sigma_{32}^2+\sigma_4^2\sigma_2^2+2\sigma_{42}^2+\sigma_3^2\sigma_4^2+2\sigma_{34}^2)]\mathbf{q}\mathbf{q}^T \end{aligned}$$

There is the expectation matrix for term 5:

$$-\frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\mathbf{q}\mathbf{q}^T\} \tag{20}$$

$$= -\tfrac{3}{2} \begin{pmatrix} 3q_1^2\sigma_1^4+q_1^2(\sigma_1^2\sigma_2^2+2\sigma_{12}^2+\sigma_1^2\sigma_3^2+2\sigma_{13}^2+\sigma_1^2\sigma_4^2+2\sigma_{14}^2) \\ 3q_2^2\sigma_2^4+q_2^2(\sigma_1^2\sigma_2^2+2\sigma_{12}^2+\sigma_2^2\sigma_3^2+2\sigma_{23}^2+\sigma_2^2\sigma_4^2+2\sigma_{24}^2) \\ 6q_1q_2(\sigma_2^2\sigma_{12}+\sigma_1^2\sigma_{12})+2q_1q_2(\sigma_3^2\sigma_{12}+2\sigma_{13}\sigma_{23}+\sigma_4^2\sigma_{12}+2\sigma_{14}\sigma_{24}) \\ 3q_3^2\sigma_3^4+q_3^2(\sigma_1^2\sigma_3^2+2\sigma_{13}^2+\sigma_2^2\sigma_3^2+2\sigma_{23}^2+\sigma_3^2\sigma_4^2+2\sigma_{34}^2) \\ 6q_1q_3(\sigma_3^2\sigma_{13}+\sigma_1^2\sigma_{13})+2q_1q_3(\sigma_2^2\sigma_{13}+2\sigma_{12}\sigma_{23}+\sigma_4^2\sigma_{13}+2\sigma_{14}\sigma_{34}) \\ 6q_2q_3(\sigma_3^2\sigma_{23}+\sigma_2^2\sigma_{23})+2q_2q_3(\sigma_1^2\sigma_{23}+2\sigma_{12}\sigma_{13}+\sigma_4^2\sigma_{23}+2\sigma_{24}\sigma_{34}) \\ 3q_4^2\sigma_4^4+q_4^2(\sigma_1^2\sigma_4^2+2\sigma_{14}^2+\sigma_2^2\sigma_4^2+2\sigma_{24}^2+\sigma_3^2\sigma_4^2+2\sigma_{34}^2) \\ 6q_1q_4(\sigma_4^2\sigma_{14}+\sigma_1^2\sigma_{14}+2q_1q_4(\sigma_2^2\sigma_{14}+2\sigma_{12}\sigma_{24}+\sigma_3^2\sigma_{14}+2\sigma_{13}\sigma_{34}) \\ 6q_2q_4(\sigma_4^2\sigma_{24}+\sigma_2^2\sigma_{24})+2q_2q_4(\sigma_1^2\sigma_{24}+2\sigma_{12}\sigma_{14}+\sigma_3^2\sigma_{24}+2\sigma_{23}\sigma_{34}) \\ 6q_3q_4(\sigma_4^2\sigma_{34}+\sigma_3^2\sigma_{34})+2q_3q_4(\sigma_1^2\sigma_{34}+2\sigma_{13}\sigma_{14}+\sigma_2^2\sigma_{34}+2\sigma_{23}\sigma_{24}) \end{pmatrix} \mathbf{q}\mathbf{q}^T$$

There is the expectation matrix for term 6:

$$
\frac{9}{4}E\{(\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q})^4\mathbf{q}\mathbf{q}^T\} \tag{21}
$$

$$
= \frac{9}{4}\begin{pmatrix}
3q_1^4\sigma_1^4 + 3q_2^4\sigma_2^4 + 12q_1q_3^3\sigma_2^2\sigma_{12} + 6q_1^2q_2^2(\sigma_1^2\sigma_2^2 + 2\sigma_{12}^2) + 12q_1^3q_2\sigma_1^2\sigma_{12} + 3q_3^4\sigma_3^4 + 12q_1q_3^2\sigma_3^2\sigma_{13} \\
12q_2q_3^3\sigma_3^2\sigma_{23} + 6q_1^2q_3^2(\sigma_1^2\sigma_3^2 + 2\sigma_{13}^2) + 6q_2^2q_3^2(\sigma_2^2\sigma_3^2 + 2\sigma_{23}^2) + 12q_1q_2q_3^2(\sigma_3^2\sigma_{12} + 2\sigma_{13}\sigma_{23}) + 12q_1^3q_3\sigma_1^2\sigma_{13} \\
12q_2^3q_3\sigma_2^2\sigma_{23} + 12q_1q_2^2q_3(\sigma_2^2\sigma_{13} + 2\sigma_{12}\sigma_{23}) + 12q_1^2q_2q_3(\sigma_1^2\sigma_{23} + 2\sigma_{12}\sigma_{13}) + 3q_4^4\sigma_4^4 + 12q_1q_4^3\sigma_4^2\sigma_{14} \\
12q_2q_4^3\sigma_4^2\sigma_{24} + 12q_3q_4^3\sigma_4^2\sigma_{34} + 6q_1^2q_4^2(\sigma_1^2\sigma_4^2 + 2\sigma_{14}^2) + 6q_2^2q_4^2(\sigma_2^2\sigma_4^2 + 2\sigma_{24}^2) + 12q_1q_2q_4^2(\sigma_4^2\sigma_{12} + 2\sigma_{14}\sigma_{24}) \\
6q_3^2q_4^2(\sigma_3^2\sigma_4^2 + 2\sigma_{34}^2) + 12q_1q_3q_4^2(\sigma_4^2\sigma_{13} + 2\sigma_{14}\sigma_{34}) + 12q_2q_3q_4^2(\sigma_4^2\sigma_{23} + 2\sigma_{24}\sigma_{34}) + 12q_1^3q_4\sigma_1^2\sigma_{14} \\
12q_2^3q_4\sigma_2^2\sigma_{24} + 12q_1q_2^2q_4(\sigma_2^2\sigma_{14} + 2\sigma_{12}\sigma_{24}) + 12q_1^2q_2q_4(\sigma_1^2\sigma_{24} + 2\sigma_{12}\sigma_{14}) + 12q_3^3q_4\sigma_3^2\sigma_{34} + 12q_1q_3^2q_4(\sigma_3^2\sigma_{14} + 2\sigma_{13}\sigma_{34}) \\
12q_2q_3^2q_4(\sigma_3^2\sigma_{24} + 2\sigma_{23}\sigma_{34}) + 12q_1^2q_3q_4(\sigma_1^2\sigma_{34} + 2\sigma_{13}\sigma_{14}) + 12q_2^2q_3q_4(\sigma_2^2\sigma_{34} + 2\sigma_{23}\sigma_{24}) + 24(\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23})
\end{pmatrix}\mathbf{q}\mathbf{q}^T
$$

4 Conclusion

This paper presents fourth-order approximate covariance matrix expressions pertaining to measurement noise, under the typical assumption of zero-mean white noise for both the body and reference vector. It offers a clear exposition of the specific numerical derivation process and outcomes. Through comparative analysis with Monte Carlo results, it is demonstrated that these expressions provide greater accuracy compared to second-order approximations while mitigating singularity issues. Furthermore, they maintain the positive definiteness of the covariance matrix's characteristic vectors.

Appendix A Proof of Eqs. (8)-(9)

The proof of Eq.(8), as follows:

$$
\begin{aligned}
P_{\boldsymbol{\Delta}\hat{\mathbf{q}}} =& E\{(N_1\boldsymbol{\Delta}\check{\mathbf{q}} + (N_2 - \bar{N}_2)\mathbf{q})(N_1\boldsymbol{\Delta}\check{\mathbf{q}} + (N_2 - \bar{N}_2)\mathbf{q})^T\} \\
=& N_1P_{\boldsymbol{\Delta}\check{\mathbf{q}}}N_1^T + E\{(N_2 - \bar{N}_2)\mathbf{q}\mathbf{q}^T(N_2 - \bar{N}_2)^T\} \\
=& N_1P_{\boldsymbol{\Delta}\check{\mathbf{q}}}N_1^T + P_{\boldsymbol{\Delta}\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^TP_{\boldsymbol{\Delta}\check{\mathbf{q}}} + \frac{1}{2}P_{\boldsymbol{\Delta}\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4 + \frac{1}{2}\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\mathbf{q}\mathbf{q}^TP_{\boldsymbol{\Delta}\check{\mathbf{q}}} \\
&+ \frac{1}{4}\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4 - E\{\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^TP_{\boldsymbol{\Delta}\check{\mathbf{q}}}\} - \frac{1}{2}E\{\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\} \\
&- \frac{1}{2}E\{[\boldsymbol{\Delta}\check{\mathbf{q}}^TM\boldsymbol{\Delta}\check{\mathbf{q}}I_4]\mathbf{q}\mathbf{q}^TP_{\boldsymbol{\Delta}\check{\mathbf{q}}}\} - \frac{1}{4}E\{[\boldsymbol{\Delta}\check{\mathbf{q}}^TM\boldsymbol{\Delta}\check{\mathbf{q}}I_4]\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\} - E\{P_{\boldsymbol{\Delta}\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\} \\
&- \frac{1}{2}E\{P_{\boldsymbol{\Delta}\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^T[\boldsymbol{\Delta}\check{\mathbf{q}}^TM\boldsymbol{\Delta}\check{\mathbf{q}}I_4]\} - \frac{1}{2}E\{\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\} + E\{\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\} \\
&- \frac{1}{4}E\{\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\} + \frac{1}{2}E\{\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\} \\
&+ \frac{1}{2}E\{\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\} + \frac{1}{4}E\{\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\} \\
=& N_1P_{\boldsymbol{\Delta}\check{\mathbf{q}}}N_1^T - E\{P_{\boldsymbol{\Delta}\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\} - \frac{1}{2}E\{P_{\boldsymbol{\Delta}\check{\mathbf{q}}}\mathbf{q}\mathbf{q}^T[\boldsymbol{\Delta}\check{\mathbf{q}}^TM\boldsymbol{\Delta}\check{\mathbf{q}}I_4]\} \\
&- \frac{1}{2}E\{\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\} - \frac{1}{4}E\{\,\mathrm{tr}(MP_{\boldsymbol{\Delta}\check{\mathbf{q}}})\,I_4\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\} \\
&+ E\{\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\} + \frac{1}{2}E\{\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\} \\
&+ \frac{1}{2}E\{\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\mathbf{q}\mathbf{q}^T\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T\} + \frac{1}{4}E\{\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(M\boldsymbol{\Delta}\check{\mathbf{q}}\boldsymbol{\Delta}\check{\mathbf{q}}^T)\,I_4\}
\end{aligned} \tag{A1}
$$

The proof of Eq.(9), as follows:

$$
\begin{aligned}
P_{\Delta\hat{\mathbf{q}}4th} &= + E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2}E\{\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\,I_4\} \\
&\quad + \frac{1}{2}E\{\,\mathrm{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\,I_4\mathbf{q}\mathbf{q}^T\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{4}E\{\,\mathrm{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\,I_4\mathbf{q}\mathbf{q}^T\,\mathrm{tr}(M\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T)\,I_4\} \\
&= + E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} + \frac{1}{2}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\} + \frac{1}{2}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\}^T \\
&\quad - \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^3\Delta\check{\mathbf{q}}\mathbf{q}^T\} - \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^3\Delta\check{\mathbf{q}}\mathbf{q}^T\}^T + \frac{1}{4}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})^2\mathbf{q}\mathbf{q}^T\} \\
&\quad - \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\mathbf{q}\mathbf{q}^T\} + \frac{9}{4}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^4\mathbf{q}\mathbf{q}^T\} \\
&= + E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} - 3E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^3\Delta\check{\mathbf{q}}\mathbf{q}^T\} + E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\mathbf{q}\mathbf{q}^T\} \\
&\quad + \frac{1}{4}E\{(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})^2\mathbf{q}\mathbf{q}^T\} - \frac{3}{2}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2(\Delta\check{\mathbf{q}}^T\Delta\check{\mathbf{q}})\mathbf{q}\mathbf{q}^T\} + \frac{9}{4}E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^4\mathbf{q}\mathbf{q}^T\}
\end{aligned}
\tag{A2}
$$

Appendix B Proof of Eqs. (16)-(21)

The proof of Eq.(16), as follows:

$$
E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^2\Delta\check{\mathbf{q}}\Delta\check{\mathbf{q}}^T\} \tag{B1}
$$

$$
= E\begin{pmatrix}
\begin{array}{l} q_1^2x_1^4 + q_2^2x_1^2x_2^2 \\ + 2q_1q_2x_1^3x_2 + q_3^2x_1^2x_3^2 \\ + 2q_1q_3x_1^3x_3 + 2q_2q_3x_1^2x_2x_3 \\ + q_4^2x_1^2x_4^2 + 2q_1q_4x_1^3x_4 \\ + 2q_2q_4x_1^2x_2x_4 + 2q_3q_4x_1^2x_3x_4 \end{array} &
\begin{array}{l} q_2^2x_1x_2^3 + 2q_1q_2x_1^2x_2^2 \\ + q_1^2x_1^3x_2 + q_3^2x_1x_2x_3^2 \\ + 2q_2q_3x_1x_2^2x_3 + 2q_1q_3x_1^2x_2x_3 \\ + q_4^2x_1x_2x_4^2 + 2q_2q_4x_1x_2^2x_4 \\ + 2q_1q_4x_1^2x_2x_4 + 2q_3q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_3^2x_1x_3^3 + 2q_1q_3x_1^2x_3^2 \\ + 2q_2q_3x_1x_2x_3^2 + q_1^2x_1^3x_3 \\ + q_2^2x_1x_2^2x_3 + 2q_1q_2x_1^2x_2x_3 \\ + q_4^2x_1x_3x_4^2 + 2q_3q_4x_1x_3^2x_4 \\ + 2q_1q_4x_1^2x_3x_4 + 2q_2q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_4^2x_1x_4^3 + 2q_1q_4x_1^2x_4^2 \\ + 2q_2q_4x_1x_2x_4^2 + 2q_3q_4x_1x_3x_4^2 \\ + q_1^2x_1^3x_4 + q_2^2x_1x_2^2x_4 \\ + 2q_1q_2x_1^2x_2x_4 + q_3^2x_1x_3^2x_4 \\ + 2q_1q_3x_1^2x_3x_4 + 2q_2q_3x_1x_2x_3x_4 \end{array} \\
\begin{array}{l} q_2^2x_1x_2^3 + 2q_1q_2x_1^2x_2^2 \\ + q_1^2x_1^3x_2 + q_3^2x_1x_2x_3^2 \\ + 2q_2q_3x_1x_2^2x_3 + 2q_1q_3x_1^2x_2x_3 \\ + q_4^2x_1x_2x_4^2 + 2q_2q_4x_1x_2^2x_4 \\ + 2q_1q_4x_1^2x_2x_4 + 2q_3q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_2^2x_2^4 + 2q_1q_2x_1x_2^3 \\ + q_1^2x_1^2x_2^2 + q_3^2x_2^2x_3^2 \\ + 2q_2q_3x_2^3x_3 + 2q_1q_3x_1x_2^2x_3 \\ + q_4^2x_2^2x_4^2 + 2q_2q_4x_2^3x_4 \\ + 2q_1q_4x_1x_2^2x_4 + 2q_3q_4x_2^2x_3x_4 \end{array} &
\begin{array}{l} q_3^2x_2x_3^3 + 2q_2q_3x_2^2x_3^2 \\ + 2q_1q_3x_1x_2x_3^2 + q_2^2x_2^3x_3 \\ + 2q_1q_2x_1x_2^2x_3 + q_1^2x_1^2x_2x_3 \\ + q_4^2x_2x_3x_4^2 + 2q_3q_4x_2x_3^2x_4 \\ + 2q_2q_4x_2^2x_3x_4 + 2q_1q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_3^2x_2x_3^3 + 2q_2q_3x_2^2x_3^2 \\ + 2q_1q_3x_1x_2x_3^2 + q_2^2x_2^3x_3 \\ + 2q_1q_2x_1x_2^2x_3 + q_1^2x_1^2x_2x_3 \\ + q_4^2x_2x_3x_4^2 + 2q_3q_4x_2x_3^2x_4 \\ + 2q_2q_4x_2^2x_3x_4 + 2q_1q_4x_1x_2x_3x_4 \end{array} \\
\begin{array}{l} q_3^2x_1x_3^3 + 2q_1q_3x_1^2x_3^2 \\ + 2q_2q_3x_1x_2x_3^2 + q_1^2x_1^3x_3 \\ + q_2^2x_1x_2^2x_3 + 2q_1q_2x_1^2x_2x_3 \\ + q_4^2x_1x_3x_4^2 + 2q_3q_4x_1x_3^2x_4 \\ + 2q_1q_4x_1^2x_3x_4 + 2q_2q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_3^2x_2x_3^3 + 2q_2q_3x_2^2x_3^2 \\ + 2q_1q_3x_1x_2x_3^2 + q_2^2x_2^3x_3 \\ + 2q_1q_2x_1x_2^2x_3 + q_1^2x_1^2x_2x_3 \\ + q_4^2x_2x_3x_4^2 + 2q_3q_4x_2x_3^2x_4 \\ + 2q_2q_4x_2^2x_3x_4 + 2q_1q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_3^2x_3^4 + 2q_1q_3x_1x_3^3 \\ + 2q_2q_3x_2x_3^3 + q_1^2x_1^2x_3^2 \\ + q_2^2x_2^2x_3^2 + 2q_1q_2x_1x_2x_3^2 \\ + q_4^2x_3^2x_4^2 + 2q_3q_4x_3^3x_4 \\ + 2q_1q_4x_1x_3^2x_4 + 2q_2q_4x_2x_3^2x_4 \end{array} &
\begin{array}{l} q_4^2x_3x_4^3 + 2q_3q_4x_3^2x_4^2 \\ + 2q_1q_4x_1x_3x_4^2 + 2q_2q_4x_2x_3x_4^2 \\ + q_3^2x_3^3x_4 + 2q_1q_3x_1x_3^2x_4 \\ + 2q_2q_3x_2x_3^2x_4 + q_1^2x_1^2x_3x_4 \\ + q_2^2x_2^2x_3x_4 + 2q_1q_2x_1x_2x_3x_4 \end{array} \\
\begin{array}{l} q_4^2x_1x_4^3 + 2q_1q_4x_1^2x_4^2 \\ + 2q_2q_4x_1x_2x_4^2 + 2q_3q_4x_1x_3x_4^2 \\ + q_1^2x_1^3x_4 + q_2^2x_1x_2^2x_4 \\ + 2q_1q_2x_1^2x_2x_4 + q_3^2x_1x_3^2x_4 \\ + 2q_1q_3x_1^2x_3x_4 + 2q_2q_3x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_4^2x_2x_4^3 + 2q_2q_4x_2^2x_4^2 \\ + 2q_1q_4x_1x_2x_4^2 + 2q_3q_4x_2x_3x_4^2 \\ + q_2^2x_2^3x_4 + 2q_1q_2x_1x_2^2x_4 \\ + q_1^2x_1^2x_2x_4 + q_3^2x_2x_3^2x_4 \\ + 2q_2q_3x_2^2x_3x_4 + 2q_1q_3x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_4^2x_3x_4^3 + 2q_3q_4x_3^2x_4^2 \\ + 2q_1q_4x_1x_3x_4^2 + 2q_2q_4x_2x_3x_4^2 \\ + q_3^2x_3^3x_4 + 2q_1q_3x_1x_3^2x_4 \\ + 2q_2q_3x_2x_3^2x_4 + q_1^2x_1^2x_3x_4 \\ + q_2^2x_2^2x_3x_4 + 2q_1q_2x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_4^2x_4^4 + 2q_1q_4x_1x_4^3 \\ + 2q_2q_4x_2x_4^3 + 2q_3q_4x_3x_4^3 \\ + q_1^2x_1^2x_4^2 + q_2^2x_2^2x_4^2 \\ + 2q_1q_2x_1x_2x_4^2 + q_3^2x_3^2x_4^2 \\ + 2q_1q_3x_1x_3x_4^2 + 2q_2q_3x_2x_3x_4^2 \end{array}
\end{pmatrix}
$$

The proof of Eq.(17), as follows:

$$
-3E\{(\Delta\check{\mathbf{q}}^T\mathbf{q})^3\Delta\check{\mathbf{q}}\mathbf{q}^T\} \tag{B2}
$$

$$
= -3E\begin{pmatrix}
\begin{array}{l} q_1^4x_1^4 + q_1q_2^3x_1x_2^3 + 3q_1^2q_2^2x_1^2x_2^2 \\ + 3q_1^3q_2x_1^3x_2 + q_1q_3^3x_1x_3^3 \\ + 3q_1^2q_3^2x_1^2x_3^2 + 3q_1q_2q_3^2x_1x_2x_3^2 \\ + 3q_1^3q_3x_1^3x_3 + 3q_1q_2^2q_3x_1x_2^2x_3 \\ + 6q_1^2q_2q_3x_1^2x_2x_3 + q_1q_4^3x_1x_4^3 \\ + 3q_1^2q_4^2x_1^2x_4^2 + 3q_1q_2q_4^2x_1x_2x_4^2 \\ + 3q_1q_3q_4^2x_1x_3x_4^2 + 3q_1^3q_4x_1^3x_4 \\ + 3q_1q_2^2q_4x_1x_2^2x_4 + 6q_1^2q_2q_4x_1^2x_2x_4 \\ + 3q_1q_3^2q_4x_1x_3^2x_4 + 6q_1^2q_3q_4x_1^2x_3x_4 \\ + 6q_1q_2q_3q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_1^3q_2x_1^4 + q_2^4x_1x_2^3 + 3q_1q_2^3x_1^2x_2^2 \\ + 3q_1^2q_2^2x_1^3x_2 + q_2q_3^3x_1x_3^3 \\ + 3q_1q_2q_3^2x_1^2x_3^2 + 3q_2^2q_3^2x_1x_2x_3^2 \\ + 3q_1^2q_2q_3x_1^3x_3 + 3q_2^3q_3x_1x_2^2x_3 \\ + 6q_1q_2^2q_3x_1^2x_2x_3 + q_2q_4^3x_1x_4^3 \\ + 3q_1q_2q_4^2x_1^2x_4^2 + 3q_2^2q_4^2x_1x_2x_4^2 \\ + 3q_2q_3q_4^2x_1x_3x_4^2 + 3q_1^2q_2q_4x_1^3x_4 \\ + 3q_2^3q_4x_1x_2^2x_4 + 6q_1q_2^2q_4x_1^2x_2x_4 \\ + 3q_2q_3^2q_4x_1x_3^2x_4 + 6q_1q_2q_3q_4x_1^2x_3x_4 \\ + 6q_2^2q_3q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_1^3q_3x_1^4 + q_2^3q_3x_1x_2^3 + 3q_1q_2^2q_3x_1^2x_2^2 \\ + 3q_1^2q_2q_3x_1^3x_2 + q_3^4x_1x_3^3 \\ + 3q_1q_3^3x_1^2x_3^2 + 3q_2q_3^3x_1x_2x_3^2 \\ + 3q_1^2q_3^2x_1^3x_3 + 3q_2^2q_3^2x_1x_2^2x_3 \\ + 6q_1q_2q_3^2x_1^2x_2x_3 + q_3q_4^3x_1x_4^3 \\ + 3q_1q_3q_4^2x_1^2x_4^2 + 3q_2q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_1x_3x_4^2 + 3q_1^2q_3q_4x_1^3x_4 \\ + 3q_2^2q_3q_4x_1x_2^2x_4 + 6q_1q_2q_3q_4x_1^2x_2x_4 \\ + 3q_3^3q_4x_1x_3^2x_4 + 6q_1q_3^2q_4x_1^2x_3x_4 \\ + 6q_2q_3^2q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_1^3q_4x_1^4 + q_2^3q_4x_1x_2^3 + 3q_1q_2^2q_4x_1^2x_2^2 \\ + 3q_1^2q_2q_4x_1^3x_2 + q_3^3q_4x_1x_3^3 \\ + 3q_1q_3^2q_4x_1^2x_3^2 + 3q_2q_3^2q_4x_1x_2x_3^2 \\ + 3q_1^2q_3q_4x_1^3x_3 + 3q_2^2q_3q_4x_1x_2^2x_3 \\ + 6q_1q_2q_3q_4x_1^2x_2x_3 + q_4^4x_1x_4^3 \\ + 3q_1q_4^3x_1^2x_4^2 + 3q_2q_4^3x_1x_2x_4^2 \\ + 3q_3q_4^3x_1x_3x_4^2 + 3q_1^2q_4^2x_1^3x_4 \\ + 3q_2^2q_4^2x_1x_2^2x_4 + 6q_1q_2q_4^2x_1^2x_2x_4 \\ + 3q_3^2q_4^2x_1x_3^2x_4 + 6q_1q_3q_4^2x_1^2x_3x_4 \\ + 6q_2q_3q_4^2x_1x_2x_3x_4 \end{array} \\
\begin{array}{l} q_1^3q_2x_1^4 + q_2^4x_1x_2^3 + 3q_1q_2^3x_1^2x_2^2 \\ + 3q_1^2q_2^2x_1^3x_2 + q_2q_3^3x_1x_3^3 \\ + 3q_1q_2q_3^2x_1^2x_3^2 + 3q_2^2q_3^2x_1x_2x_3^2 \\ + 3q_1^2q_2q_3x_1^3x_3 + 3q_2^3q_3x_1x_2^2x_3 \\ + 6q_1q_2^2q_3x_1^2x_2x_3 + q_2q_4^3x_1x_4^3 \\ + 3q_1q_2q_4^2x_1^2x_4^2 + 3q_2^2q_4^2x_1x_2x_4^2 \\ + 3q_2q_3q_4^2x_1x_3x_4^2 + 3q_1^2q_2q_4x_1^3x_4 \\ + 3q_2^3q_4x_1x_2^2x_4 + 6q_1q_2^2q_4x_1^2x_2x_4 \\ + 3q_2q_3^2q_4x_1x_3^2x_4 + 6q_1q_2q_3q_4x_1^2x_3x_4 \\ + 6q_2^2q_3q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_2^4x_2^4 + 3q_1q_2^3x_1x_2^3 + 3q_1^2q_2^2x_1^2x_2^2 \\ + q_1^3q_2x_1^3x_2 + q_2q_3^3x_2x_3^3 \\ + 3q_2^2q_3^2x_2^2x_3^2 + 3q_1q_2q_3^2x_1x_2x_3^2 \\ + 3q_2^3q_3x_2^3x_3 + 6q_1q_2^2q_3x_1x_2^2x_3 \\ + 3q_1^2q_2q_3x_1^2x_2x_3 + q_2q_4^3x_2x_4^3 \\ + 3q_2^2q_4^2x_2^2x_4^2 + 3q_1q_2q_4^2x_1x_2x_4^2 \\ + 3q_2q_3q_4^2x_2x_3x_4^2 + 3q_2^3q_4x_2^3x_4 \\ + 6q_1q_2^2q_4x_1x_2^2x_4 + 3q_1^2q_2q_4x_1^2x_2x_4 \\ + 3q_2q_3^2q_4x_2x_3^2x_4 + 6q_2^2q_3q_4x_2^2x_3x_4 \\ + 6q_1q_2q_3q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_2^3q_3x_2^4 + 3q_1q_2^2q_3x_1x_2^3 + 3q_1^2q_2q_3x_1^2x_2^2 \\ + q_1^3q_3x_1^3x_2 + q_3^4x_2x_3^3 \\ + 3q_2q_3^3x_2^2x_3^2 + 3q_1q_3^3x_1x_2x_3^2 \\ + 3q_2^2q_3^2x_2^3x_3 + 6q_1q_2q_3^2x_1x_2^2x_3 \\ + 3q_1^2q_3^2x_1^2x_2x_3 + q_3q_4^3x_2x_4^3 \\ + 3q_2q_3q_4^2x_2^2x_4^2 + 3q_1q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_2x_3x_4^2 + 3q_2^2q_3q_4x_2^3x_4 \\ + 6q_1q_2q_3q_4x_1x_2^2x_4 + 3q_1^2q_3q_4x_1^2x_2x_4 \\ + 3q_3^3q_4x_2x_3^2x_4 + 6q_2q_3^2q_4x_2^2x_3x_4 \\ + 6q_1q_3^2q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_2^3q_4x_2^4 + 3q_1q_2^2q_4x_1x_2^3 + 3q_1^2q_2q_4x_1^2x_2^2 \\ + q_1^3q_4x_1^3x_2 + q_3^3q_4x_2x_3^3 \\ + 3q_2q_3^2q_4x_2^2x_3^2 + 3q_1q_3^2q_4x_1x_2x_3^2 \\ + 3q_2^2q_3q_4x_2^3x_3 + 6q_1q_2q_3q_4x_1x_2^2x_3 \\ + 3q_1^2q_3q_4x_1^2x_2x_3 + q_4^4x_2x_4^3 \\ + 3q_2q_4^3x_2^2x_4^2 + 3q_1q_4^3x_1x_2x_4^2 \\ + 3q_3q_4^3x_2x_3x_4^2 + 3q_2^2q_4^2x_2^3x_4 \\ + 6q_1q_2q_4^2x_1x_2^2x_4 + 3q_1^2q_4^2x_1^2x_2x_4 \\ + 3q_3^2q_4^2x_2x_3^2x_4 + 6q_2q_3q_4^2x_2^2x_3x_4 \\ + 6q_1q_3q_4^2x_1x_2x_3x_4 \end{array} \\
\begin{array}{l} q_1^3q_3x_1^4 + q_2^3q_3x_1x_2^3 + 3q_1q_2^2q_3x_1^2x_2^2 \\ + 3q_1^2q_2q_3x_1^3x_2 + q_3^4x_1x_3^3 \\ + 3q_1q_3^3x_1^2x_3^2 + 3q_2q_3^3x_1x_2x_3^2 \\ + 3q_1^2q_3^2x_1^3x_3 + 3q_2^2q_3^2x_1x_2^2x_3 \\ + 6q_1q_2q_3^2x_1^2x_2x_3 + q_3q_4^3x_1x_4^3 \\ + 3q_1q_3q_4^2x_1^2x_4^2 + 3q_2q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_1x_3x_4^2 + 3q_1^2q_3q_4x_1^3x_4 \\ + 3q_2^2q_3q_4x_1x_2^2x_4 + 6q_1q_2q_3q_4x_1^2x_2x_4 \\ + 3q_3^3q_4x_1x_3^2x_4 + 6q_1q_3^2q_4x_1^2x_3x_4 \\ + 6q_2q_3^2q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_2^3q_3x_2^4 + 3q_1q_2^2q_3x_1x_2^3 + 3q_1^2q_2q_3x_1^2x_2^2 \\ + q_1^3q_3x_1^3x_2 + q_3^4x_2x_3^3 \\ + 3q_2q_3^3x_2^2x_3^2 + 3q_1q_3^3x_1x_2x_3^2 \\ + 3q_2^2q_3^2x_2^3x_3 + 6q_1q_2q_3^2x_1x_2^2x_3 \\ + 3q_1^2q_3^2x_1^2x_2x_3 + q_3q_4^3x_2x_4^3 \\ + 3q_2q_3q_4^2x_2^2x_4^2 + 3q_1q_3q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_2x_3x_4^2 + 3q_2^2q_3q_4x_2^3x_4 \\ + 6q_1q_2q_3q_4x_1x_2^2x_4 + 3q_1^2q_3q_4x_1^2x_2x_4 \\ + 3q_3^3q_4x_2x_3^2x_4 + 6q_2q_3^2q_4x_2^2x_3x_4 \\ + 6q_1q_3^2q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_3^4x_3^4 + 3q_1q_3^3x_1x_3^3 + 3q_2q_3^3x_2x_3^3 \\ + 3q_1^2q_3^2x_1^2x_3^2 + 3q_2^2q_3^2x_2^2x_3^2 \\ + 6q_1q_2q_3^2x_1x_2x_3^2 + q_1^3q_3x_1^3x_3 \\ + q_2^3q_3x_2^3x_3 + 3q_1q_2^2q_3x_1x_2^2x_3 \\ + 3q_1^2q_2q_3x_1^2x_2x_3 + q_3q_4^3x_3x_4^3 \\ + 3q_3^2q_4^2x_3^2x_4^2 + 3q_1q_3q_4^2x_1x_3x_4^2 \\ + 3q_2q_3q_4^2x_2x_3x_4^2 + 3q_3^3q_4x_3^3x_4 \\ + 6q_1q_3^2q_4x_1x_3^2x_4 + 6q_2q_3^2q_4x_2x_3^2x_4 \\ + 3q_1^2q_3q_4x_1^2x_3x_4 + 3q_2^2q_3q_4x_2^2x_3x_4 \\ + 6q_1q_2q_3q_4x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_3^3q_4x_3^4 + 3q_1q_3^2q_4x_1x_3^3 + 3q_2q_3^2q_4x_2x_3^3 \\ + 3q_1^2q_3q_4x_1^2x_3^2 + 3q_2^2q_3q_4x_2^2x_3^2 \\ + 6q_1q_2q_3q_4x_1x_2x_3^2 + q_1^3q_4x_1^3x_3 \\ + q_2^3q_4x_2^3x_3 + 3q_1q_2^2q_4x_1x_2^2x_3 \\ + 3q_1^2q_2q_4x_1^2x_2x_3 + q_4^4x_3x_4^3 \\ + 3q_3q_4^3x_3^2x_4^2 + 3q_1q_4^3x_1x_3x_4^2 \\ + 3q_2q_4^3x_2x_3x_4^2 + 3q_3^2q_4^2x_3^3x_4 \\ + 6q_1q_3q_4^2x_1x_3^2x_4 + 6q_2q_3q_4^2x_2x_3^2x_4 \\ + 3q_1^2q_4^2x_1^2x_3x_4 + 3q_2^2q_4^2x_2^2x_3x_4 \\ + 6q_1q_2q_4^2x_1x_2x_3x_4 \end{array} \\
\begin{array}{l} q_1^3q_4x_1^4 + q_2^3q_4x_1x_2^3 + 3q_1q_2^2q_4x_1^2x_2^2 \\ + 3q_1^2q_2q_4x_1^3x_2 + q_3^3q_4x_1x_3^3 \\ + 3q_1q_3^2q_4x_1^2x_3^2 + 3q_2q_3^2q_4x_1x_2x_3^2 \\ + 3q_1^2q_3q_4x_1^3x_3 + 3q_2^2q_3q_4x_1x_2^2x_3 \\ + 6q_1q_2q_3q_4x_1^2x_2x_3 + q_4^4x_1x_4^3 \\ + 3q_1q_4^3x_1^2x_4^2 + 3q_2q_4^3x_1x_2x_4^2 \\ + 3q_3q_4^3x_1x_3x_4^2 + 3q_1^2q_4^2x_1^3x_4 \\ + 3q_2^2q_4^2x_1x_2^2x_4 + 6q_1q_2q_4^2x_1^2x_2x_4 \\ + 3q_3^2q_4^2x_1x_3^2x_4 + 6q_1q_3q_4^2x_1^2x_3x_4 \\ + 6q_2q_3q_4^2x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_2^3q_4x_2^4 + 3q_1q_2^2q_4x_1x_2^3 + 3q_1^2q_2q_4x_1^2x_2^2 \\ + q_1^3q_4x_1^3x_2 + q_3^3q_4x_2x_3^3 \\ + 3q_2q_3^2q_4x_2^2x_3^2 + 3q_1q_3^2q_4x_1x_2x_3^2 \\ + 3q_2^2q_3q_4x_2^3x_3 + 6q_1q_2q_3q_4x_1x_2^2x_3 \\ + 3q_1^2q_3q_4x_1^2x_2x_3 + q_4^4x_2x_4^3 \\ + 3q_2q_4^3x_2^2x_4^2 + 3q_1q_4^3x_1x_2x_4^2 \\ + 3q_3q_4^3x_2x_3x_4^2 + 3q_2^2q_4^2x_2^3x_4 \\ + 6q_1q_2q_4^2x_1x_2^2x_4 + 3q_1^2q_4^2x_1^2x_2x_4 \\ + 3q_3^2q_4^2x_2x_3^2x_4 + 6q_2q_3q_4^2x_2^2x_3x_4 \\ + 6q_1q_3q_4^2x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_3^3q_4x_3^4 + 3q_1q_3^2q_4x_1x_3^3 + 3q_2q_3^2q_4x_2x_3^3 \\ + 3q_1^2q_3q_4x_1^2x_3^2 + 3q_2^2q_3q_4x_2^2x_3^2 \\ + 6q_1q_2q_3q_4x_1x_2x_3^2 + q_1^3q_4x_1^3x_3 \\ + q_2^3q_4x_2^3x_3 + 3q_1q_2^2q_4x_1x_2^2x_3 \\ + 3q_1^2q_2q_4x_1^2x_2x_3 + q_4^4x_3x_4^3 \\ + 3q_3q_4^3x_3^2x_4^2 + 3q_1q_4^3x_1x_3x_4^2 \\ + 3q_2q_4^3x_2x_3x_4^2 + 3q_3^2q_4^2x_3^3x_4 \\ + 6q_1q_3q_4^2x_1x_3^2x_4 + 6q_2q_3q_4^2x_2x_3^2x_4 \\ + 3q_1^2q_4^2x_1^2x_3x_4 + 3q_2^2q_4^2x_2^2x_3x_4 \\ + 6q_1q_2q_4^2x_1x_2x_3x_4 \end{array} &
\begin{array}{l} q_4^4x_4^4 + 3q_1q_4^3x_1x_4^3 + 3q_2q_4^3x_2x_4^3 \\ + 3q_3q_4^3x_3x_4^3 + 3q_1^2q_4^2x_1^2x_4^2 \\ + 3q_2^2q_4^2x_2^2x_4^2 + 6q_1q_2q_4^2x_1x_2x_4^2 \\ + 3q_3^2q_4^2x_3^2x_4^2 + 6q_1q_3q_4^2x_1x_3x_4^2 \\ + 6q_2q_3q_4^2x_2x_3x_4^2 + q_1^3q_4x_1^3x_4 \\ + q_2^3q_4x_2^3x_4 + 3q_1q_2^2q_4x_1x_2^2x_4 \\ + 3q_1^2q_2q_4x_1^2x_2x_4 + q_3^3q_4x_3^3x_4 \\ + 3q_1q_3^2q_4x_1x_3^2x_4 + 3q_2q_3^2q_4x_2x_3^2x_4 \\ + 3q_1^2q_3q_4x_1^2x_3x_4 + 3q_2^2q_3q_4x_2^2x_3x_4 \\ + 6q_1q_2q_3q_4x_1x_2x_3x_4 \end{array}
\end{pmatrix}
$$

The proof of Eq.(18), as follows:

$$E\{(\mathbf{\Delta\check{q}}^T\mathbf{\Delta\check{q}})\mathbf{\Delta\check{q}}\mathbf{\Delta\check{q}}^T\mathbf{q}\mathbf{q}^T\} \tag{B3}$$

$$
= E\begin{pmatrix}
\begin{array}{l} x_1^4 + x_1^2x_2^2 \\ + x_1^2x_3^2 + x_1^2x_4^2 \end{array} &
\begin{array}{l} x_1x_2^3 + x_1^3x_2 \\ + x_1x_2x_3^2 + x_1x_2x_4^2 \end{array} &
\begin{array}{l} x_1x_3^3 + x_1^3x_3 \\ + x_1x_2^2x_3 + x_1x_3x_4^2 \end{array} &
\begin{array}{l} x_1x_4^3 + x_1^3x_4 \\ + x_1x_2^2x_4 + x_1x_3^2x_4 \end{array} \\
\begin{array}{l} x_1x_2^3 + x_1^3x_2 \\ + x_1x_2x_3^2 + x_1x_2x_4^2 \end{array} &
\begin{array}{l} x_2^4 + x_1^2x_2^2 \\ + x_2^2x_3^2 + x_2^2x_4^2 \end{array} &
\begin{array}{l} x_2x_3^3 + x_2^3x_3 \\ + x_1^2x_2x_3 + x_2x_3x_4^2 \end{array} &
\begin{array}{l} x_2x_4^3 + x_2^3x_4 \\ + x_1^2x_2x_4 + x_2x_3^2x_4 \end{array} \\
\begin{array}{l} x_1x_3^3 + x_1^3x_3 \\ + x_1x_2^2x_3 + x_1x_3x_4^2 \end{array} &
\begin{array}{l} x_2x_3^3 + x_2^3x_3 \\ + x_1^2x_2x_3 + x_2x_3x_4^2 \end{array} &
\begin{array}{l} x_3^4 + x_1^2x_3^2 \\ + x_2^2x_3^2 + x_3^2x_4^2 \end{array} &
\begin{array}{l} x_3x_4^3 + x_3^3x_4 \\ + x_1^2x_3x_4 + x_2^2x_3x_4 \end{array} \\
\begin{array}{l} x_1x_4^3 + x_1^3x_4 \\ + x_1x_2^2x_4 + x_1x_3^2x_4 \end{array} &
\begin{array}{l} x_2x_4^3 + x_2^3x_4 \\ + x_1^2x_2x_4 + x_2x_3^2x_4 \end{array} &
\begin{array}{l} x_3x_4^3 + x_3^3x_4 \\ + x_1^2x_3x_4 + x_2^2x_3x_4 \end{array} &
\begin{array}{l} x_4^4 + x_1^2x_4^2 \\ + x_2^2x_4^2 + x_3^2x_4^2 \end{array}
\end{pmatrix}\mathbf{q}\mathbf{q}^T
$$

The proof of Eq.(19), as follows:

$$\frac{1}{4}E\{(\mathbf{\Delta\check{q}}^T\mathbf{\Delta\check{q}})^2\mathbf{q}\mathbf{q}^T\} \tag{B4}$$

$$= \frac{1}{4}E\{x_1^4 + x_2^4 + 2x_1^2x_2^2 + x_3^4 + 2x_1^2x_3^2 + 2x_2^2x_3^2 + x_4^4 + 2x_1^2x_4^2 + 2x_2^2x_4^2 + 2x_3^2x_4^2\}\mathbf{q}\mathbf{q}^T$$

The proof of Eq.(20), as follows:

$$-\frac{3}{2}E\{(\mathbf{\Delta\check{q}}^T\mathbf{q})^2(\mathbf{\Delta\check{q}}^T\mathbf{\Delta\check{q}})\mathbf{q}\mathbf{q}^T\} \tag{B5}$$

$$= -\tfrac{3}{2}E\begin{pmatrix} q_1^2x_1^4 + q_2^2x_2^4 + 2q_1q_2x_1x_2^3 + q_1^2x_1^2x_2^2 + q_2^2x_1^2x_2^2 \\ + 2q_1q_2x_1^3x_2 + q_3^2x_3^4 + 2q_1q_3x_1x_3^3 + 2q_2q_3x_2x_3^3 + q_1^2x_1^2x_3^2 \\ + q_3^2x_1^2x_3^2 + q_2^2x_2^2x_3^2 + q_3^2x_2^2x_3^2 + 2q_1q_2x_1x_2x_3^2 + 2q_1q_3x_1^3x_3 \\ + 2q_2q_3x_2^3x_3 + 2q_1q_3x_1x_2^2x_3 + 2q_2q_3x_1^2x_2x_3 + q_4^2x_4^4 + 2q_1q_4x_1x_4^3 \\ + 2q_2q_4x_2x_4^3 + 2q_3q_4x_3x_4^3 + q_1^2x_1^2x_4^2 + q_4^2x_1^2x_4^2 + q_2^2x_2^2x_4^2 \\ + q_4^2x_2^2x_4^2 + 2q_1q_2x_1x_2x_4^2 + q_3^2x_3^2x_4^2 + q_4^2x_3^2x_4^2 \\ + 2q_1q_3x_1x_3x_4^2 + 2q_2q_3x_2x_3x_4^2 + 2q_1q_4x_1^3x_4 + 2q_2q_4x_2^3x_4 \\ + 2q_1q_4x_1x_2^2x_4 + 2q_2q_4x_1^2x_2x_4 + 2q_3q_4x_3^3x_4 + 2q_1q_4x_1x_3^2x_4 \\ + 2q_2q_4x_2x_3^2x_4 + 2q_3q_4x_1^2x_3x_4 + 2q_3q_4x_2^2x_3x_4 \end{pmatrix}\mathbf{q}\mathbf{q}^T$$

The proof of Eq.(21), as follows:

$$\frac{9}{4}E\{(\mathbf{\Delta\check{q}}^T\mathbf{q})^4\mathbf{q}\mathbf{q}^T\} \tag{B6}$$

$$= \tfrac{9}{4}E\begin{pmatrix} q_1^4x_1^4 + q_2^4x_2^4 + 4q_1q_2^3x_1x_2^3 + 6q_1^2q_2^2x_1^2x_2^2 \\ + 4q_1^3q_2x_1^3x_2 + q_3^4x_3^4 + 4q_1q_3^3x_1x_3^3 + 4q_2q_3^3x_2x_3^3 \\ + 6q_1^2q_3^2x_1^2x_3^2 + 6q_2^2q_3^2x_2^2x_3^2 + 12q_1q_2q_3^2x_1x_2x_3^2 \\ + 4q_1^3q_3x_1^3x_3 + 4q_2^3q_3x_2^3x_3 + 12q_1q_2^2q_3x_1x_2^2x_3 \\ + 12q_1^2q_2q_3x_1^2x_2x_3 + q_4^4x_4^4 + 4q_1q_4^3x_1x_4^3 + 4q_2q_4^3x_2x_4^3 \\ + 4q_3q_4^3x_3x_4^3 + 6q_1^2q_4^2x_1^2x_4^2 + 6q_2^2q_4^2x_2^2x_4^2 \\ + 12q_1q_1q_2q_4^2x_1x_2x_4^2 + 6q_3 + 6q_3^2q_4^2x_3^2x_4^2 \\ + 12q_1q_3q_4^2x_1x_3x_4^2 + 12q_2q_3q_4^2x_2x_3x_4^2 + 4q_1^3q_4x_1^3x_4 \\ + 4q_2^3q_4x_2x_4 + 12q_1q_2^2q_4x_1x_2^2x_4 + 12q_1^2q_2q_4x_1^2x_2x_4 \\ + 4q_3^3q_4x_3^3x_4 + 12q_3q_3^2q_4x_1x_3^2x_4 + 12q_2q_3^2q_4x_2x_3^2x_4 \\ + 12q_1^2q_3q_4x_1^2x_3x_4 + 12q_2^2q_3q_4x_2^2x_3x_4 \\ + 24q_1q_2q_3q_4x_1x_2x_3x_4 \end{pmatrix}\mathbf{q}\mathbf{q}^T$$

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